**Lecture-4**

**Complex Numbers**

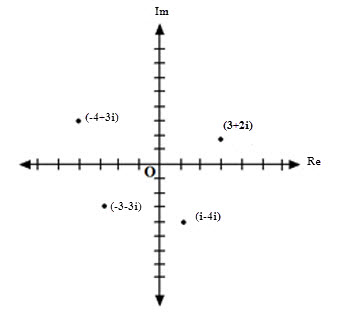
A complex number is the form  where and are real numbers and is called the imaginary unit, has the property that or . In general, if is any positive number, we would write:

If , then is called the real part of and is called the imaginary part of *z* and are denoted by **** and  respectively. From this, it is obvious that two complex numbers and are equal if and only if and, that is, the real and imaginary components are equal. If the number is said to be purely imaginary, if the number is real.

The standard rectangular form of a complex number is. The symbol , which can stand for any of complex numbers, is called a complex variable.

**Graphical Representation of Complex Number/ Argand Diagram:**

Since a complex number  can be considered as an ordered pair of real numbers, we can represent such numbers by points in a plane called the **complex plane or Argand diagram**. Mathematician Argand represented a complex number in a diagram known as **Argand diagram**.



**Figure:1**

A complex number  can be represented by a point *P* whose co-ordinates are .The horizontal axis is called the **real axis** and the vertical axis is called the **imaginary** axis.

To each complex num­ber there corresponds one and only one point in the plane, and conversely to each point in the plane there corresponds one and only one complex number. Because of this we often refer to the complex number as the point. This is shown in **Figure: 1.**

**Fundamental operations with complex number:**

**Addition and Subtraction:**

The **sum** and **difference** of complex numbers is defined by adding or subtracting their real components where i.e.:

****

For instance: ** (**similar to vector addition)

**Product:**

The commutative and distributive properties hold for the **product** of complex numbers i.e.,

****

We know: ****

Therefore giving us: ****

**Division:**

****

Basically, when **dividing** two complex numbers we are rationalizing the denominator of a rational expression multiplying the numerator and denominator by the conjugate of the denominator.

**Example:** Express  in the form 

**Solution:** We must multiply the numerator and denominator by the conjugate of  i.e.,

.

**.**

**Conjugates**

The complex conjugate, or briefly conjugate, of a complex number is. The complex conjugate of a complex number is often indicated by. The geometric interpretation of a complex conjugate is the reflection along the real axis. This can be seen in the **Figure:2** below where is a complex number. Listed below are also several properties of conjugates.

|  |  |
| --- | --- |
| **Some Properties:** | **Figure:2** |

**Absolute value/Modulus**

The distance from the origin to any complex number is the **absolute value** or **modulus.** Looking at the **Figure: 3** below we can see that Pythagoras' Theorem gives us a formula to calculate the absolute value of a complex number denoted by mod *z* or 

i.e.

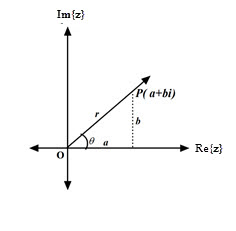
|  |
| --- |
| **Example:**  If and then evaluate. |
| **Solution:**  Given,        .  **Powers of imaginary unit**  Power of imaginary unit are given below:      One can prove by induction that for any positive integer    Hencefor all integer. If is a negative integer, we have    Example: |

**MATLAB command for complex numbers**

If  evaluate by MATLAB commands

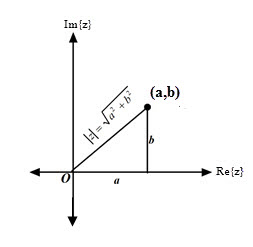
(i) (ii) (iii)  (iv)(v) 

|  |  |
| --- | --- |
| **>> clear**  **>> z1=complex(1,-1)**  **>>z1 =1.0000 - 1.0000i**  **>> z2=complex(-2, 4);**  **>> z3=complex(sqrt(3),-2);**  **>> e1=z1^2+2\*z1-3**  **e1 = -1.0000 - 4.0000i**  **>> e2=abs(2\*z2-3\*z1)**  **e2 = 13.0384**  **>> e3=abs(z1\*conj(z2)+z2\*conj(z1))**  **e3 = 12**  **>> e4=real(2\*z1^3+3\*z2^2)**  **e4 = -40**  **>> e5=imag(z1^2/z3)**  **e5 = -0.4949** | **>> clear**  **>> z1=1-i;**  **>> z2=-2+4\*i;**  **>> z3=sqrt(3)-2\*i;**  **>> e1=z1^2+2\*z1-3**  **e1 = -1.0000 - 4.0000i**  **>> e2=abs(2\*z2-3\*z1)**  **e2 = 13.0384**  **>> e3=abs(z1\*conj(z2)+z2\*conj(z1))**  **e3 = 12**  **>> e4=real(2\*z1^3+3\*z2^2)**  **e4 = -40**  **>> e5=imag(z1^2/z3)**  **e5 = -0.4949** |

  
**Figure:3**

**Polar form of Complex Number and Argument:**

It is often useful to exchange Cartesian co-ordinates to polar coordinates 



**Figure: 4**

If*P* is a point in the complex plane corre­sponding to the complex number or, then we see from **Figure: 4** that

,  (1)

where,  is the distance of the pointfrom the origin and called the **modulus or absolute value** of, denoted by mod *z* or  .

On the other hand, if , then any number *θ* satisfying the equations (1) is called an **amplitude or argument** of , and denoted by . Hence, we can write in polar form as



Note that, for a given complex number, is not unique. Since adding or subtracting multiples of  from  will result in the arm in Fig. 01 being in the same position, the argument can have many values.

Any particular choice of length , decided upon in advance, is called the **principle range** and the value of *θ* is called its principle value.

**In our study, we will consider the real number *θ* the principal argument of *z* if *θ* satisfies the equations *0 ≤ θ < 2π*** and**. The principal argument of is usually denoted by*.***

Generally, we use the formula for argument is

where may be calculated by the following formula where the quadrant containing the point corresponding to must be specified,

**N.B.** Angles measured in an anticlockwise sense are regarded as positive while those measured in a clockwise sense are regarded as negative.

It follows that

, which is called the polar form of the complex number, and and are called the polar coordinates.

From the Euler’s formula we know, .

Hence, the standard polar form of a complex number is .

**Some important properties of modulus:**

|  |  |
| --- | --- |
| **Some important properties of argument**  **Example:**  Find, where . Hence convert the number from rectangular form to polar form.  **Solution:**  Given, | |
| and .  Now**,**  **= .**        Therefore**,** the polar form of |  |

**Figure: 5**

**Example:**

Find the rectangular form of.

**Solution:**

Given,

Here and

We know that,

And

Hence .

**MATLAB command for complex numbers**

|  |  |
| --- | --- |
| **Express the following complex numbers in polar form and locate the points in complex plane:**  **>> z1=2+2\*sqrt(3)\*i;**  **>> R=abs(z1)**  **R = 4.0000**  **>> theta=angle(sym(z1))**  **theta = pi/3**  **>> z2=R\*exp(i\*theta)**  **z2 =2 + 3^(1/2)\*2i**  **>> plot(z1, '\*')** | Find rectangular form of  **>> clear**  **>> z1=sqrt(2)\*exp(i\*pi/4)**  **z1 = 1.0000 + 1.0000i** |

**Example:**

Find the principal argument of

**Solution:**

Let, , where

[

]

[

]

[

]

Therefore,

**Example:** Find the Principal argument of and

**Solution:**

Given, or,

.

**De`Moivre's Theorem**

De`Moivre's Theorem is a generalized formula to compute powers of a complex number in it's polar form.

Looking at we can find easily:

Which brings us to **De`Moivre's Theorem**:  
If and n are positive integers then

Basically, in order to find the nth power of a complex number we take the power of the absolute value or length and multiply the argument by.

**For finding *m*-th roots of a complex number:**

We can also write,

Similarly,

where,

**Euler's Form:**

We can also define this formula according to Euler’s formulae as

|  |  |
| --- | --- |
| **Example:** Find all values of for which and also locate these values in the complex plane. | |
| **Solution:** Given, .  Here the numbers of roots are 3.    or,  or,  or, [As ]  or,  when  when  when  The distance of each root from the origin is same as and the angular distance of two consecutive roots are same. | **Figure: 6** |

**MATLAB command for complex numbers :**

Find all the values of and locate the values in the complex plane

1. , (ii) , (iii) , (iv) , (v) .

|  |  |
| --- | --- |
| **>> p=[1 0 0 i];**  **>> r=roots(p)**  **r =**  **-0.8660 - 0.5000i**  **-0.0000 + 1.0000i**  **0.8660 - 0.5000i**  **>> plot(r, 'o')** | **>> p=[1 0 0 64];**  **>> r=roots(p)**  **r =**  **-4.0000 + 0.0000i**  **2.0000 + 3.4641i**  **2.0000 - 3.4641i**  **>> plot(r,'\*')** |

**Example:**

Describe and graph the locus represented by 

|  |  |
| --- | --- |
| **Solution:**  Given, .  or,  or,  or,  or, | **Figure: 7** |

**Chapter 04**

**Exercise Set**

1. Express in the form 

2. Evaluate each of the followings:

(a) , (b) , (c)

3. Convert the following numbers into polar form:

(a) (b) (c) and (d)

4. Convert the following numbers into rectangular form:

(a) and (b) .

5. Find the principal argument of the followings:

(a) (b) , (c) 

6. Find all values of for the following equations and also locate these values in the complex

plane:

(a)

(b)

(c)

(d)

7. Describe and graph the locus represented by each of the followings:

(a)

(b)

(c)

(d)

(e)

(f)